Permutations

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Contents

Permutations are one of the most interesting and useful objects in combinatorics. Part of why permutations are so useful is the flexibility they offer. We can count properties of permutations using combinatorial proofs. We can find bijections from sets we want to study to sets of permutations, and use the permutations to answer whatever questions we may have. In this section, we will look at different permutation properties, proofs that use properties of permutations, and pattern avoidance.

Expectations: depending on comfort levels, anywhere from 50-100% of this excercise set could be done with me rather than on your own. Most of the answers to these problems are available online, but it would be more enlightening to have a conversation with me rather than search out those answers on your own. The goal of this document is to figure out what you know and don't know, and what you find enjoyable or really challenging.

1 Introduction to Permutations

You may or may not have seen permutations in your math classes before now. Either way, we will have a quick refresher on the basics.

Definition 1.1. Let A be a finite set containing n elements. We define S_A as the set of bijections from A to itself. For simplicity's sake, we write this as $A = \{1, 2, \ldots, n\}$, or $A = [n]$, and use the notation S_n. This is a group under composition of functions, and the elements in this set are called permutations.

There are many different ways to write permutations. We will go over a few of those in the next example:

Example 1.1. We can consider a permutation as a **two line array**, where the image of element $j \in [n]$ appears directly below it. For example,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 6 & 5 & 1 \end{pmatrix}$$

The first line in the array is always the same, 1 up to n. We can write the same permutation in **one line notation**:

$$\sigma = 432651$$

We can also look at this function in **cycle notation**,

$$\sigma = (1\ 6)(2\ 3)(1\ 4)(5)$$

and as a product of **disjoint cycles**

$$\sigma = (1\ 4\ 6)(2\ 3)(5)$$

In general, we will write functions in S_n using Greek letters like σ, and we will write the one line notation for such a permutation as $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$.

Problem 1.1. If the permutations below are written in one line notation, write the permutation as a product of disjoint cycles. If the permutation is written as a product of cycles, write the one line form of the function, and specify which S_n you want to consider for the permutation.

1. $\alpha = 5\ 4\ 3\ 2\ 1$
2. $\beta = (1\ 3)(2\ 4\ 6\ 5)$
3. $\gamma = (1\ 2)(2\ 3)(4\ 5)(1\ 2)(5\ 6)$

Problem 1.2. Based on the one line notation for permutations, we would like to argue that there are exactly $n!$ permutations in S_n.

Definition 1.2. Let $\sigma \in S_n$. Then there exists a permutation $\tau \in S_n$ such that $\tau\sigma = Id = \sigma\tau$, and we will denote this permutation as σ^{-1}.

Example 1.2. When it comes to finding the inverse of σ, it can be easier to work with cycle notation. Consider for example, $\sigma = (1\ 4\ 6)(2\ 3)(5)$

For the next two definitions, the one line notation for the permutation is more useful than the cycles.

Definition 1.3. For $\sigma \in S_n$, written in one line notation as $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, we define the **descent set of** σ as follows: $D(\sigma) = \{i \in [n] \mid \sigma_i > \sigma_{i+1}\}$.

Descents are easy to read off of the one line version of the permutation, no matter what n is. The next definition takes a bit more work for large values of n:

Definition 1.4. For $\sigma \in S_n$, written in one line notation as $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, we define an **inversion** (sometimes **left inversions**) of σ as a pair (σ_i, σ_j) where $i < j$ and $\sigma_i > \sigma_j$. The **inversion number** of σ, denoted inv(σ) is the size of the set containing all such inversion pairs.

Example 1.3. Consider $\sigma = 432651$. We want to find $D(\sigma)$ and inv(σ).

Definition 1.5. The **major index** of σ, denoted maj(σ) is a permutation statistic defined as

$$\text{maj}(\sigma) = \sum_{i \in D(\sigma)} i$$

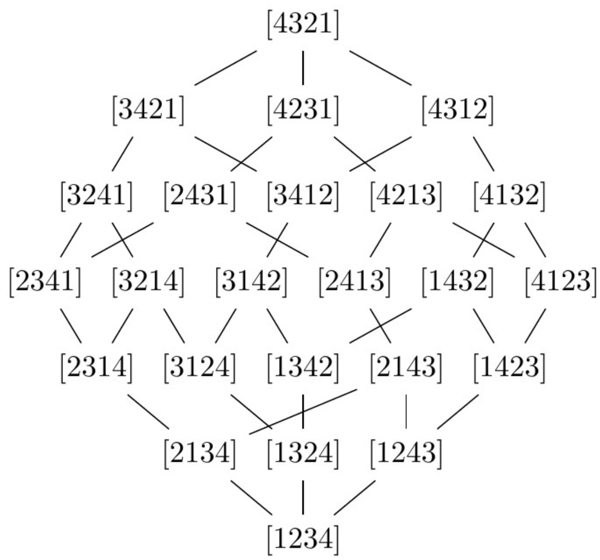
Definition 1.6. The **Lehmer code** of σ, denoted $L(\sigma)$ is an integer sequence $(L(\sigma_1), L(\sigma_2), \ldots L(\sigma_n))$, where $L(\sigma_i) = |\{j > i \mid \sigma_i > \sigma_j\}|$. Each number in the code is bounded: $0 \leq L(\sigma_i) \leq n - i$.

Problem 1.3. Consider $\sigma = 432651$. We want to find the major index and the Lehmer code for this permutation.

Inversions also allow us to define something called **a partial order for S_n**.

Definition 1.7. The **right weak order poset** is a partial order defined on S_n, denoted $W(S_n)$. This partial order is a lattice for all $n \in \mathbb{N}$
The cover relations are defined on the addition or removal of a single inversion pair.

Below is the lattice for S_4:



There are many more statistics on permutations that we have not discussed here. During the last two years, much of my research has been on permutation statistics and other related object statistics [4]. For more information, you can go to

```
http://www.findstat.org/
```

and explore the database of existing statistics. FindStat also includes Sage code for each statistic as part of the database entry.

If you want a list of commands in one place, you can go to

```
https://doc.sagemath.org/html/en/reference/combinat/sage/combinat/permutation.html
```

For example, to find all the inversion pairs for $\sigma = 76124385$, you can save a lot of time by running the code

```
Permutation([7,6,1,2,4,3,8,5]).inversions()
```

in Sage.

This is where I would like to spend a little bit of time exploring things. What permutation statistics are "easy" to spot in the matrix forms?

2 Fun with Permutations

Pick one of the three examples in this section to play with. You need not complete all of them, this is just to show how varied the applications of permutations are.

Definition 2.1. The number $c(n,k)$ is defined as the number of permutations in S_n with exactly k cycles.

Lemma 2.1. The numbers $c(n,k)$ satisfy the recurrence

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$$

for $n, k \geq 1$, with the initial conditions $c(n,k) = 0$ if $n < k$ or $k = 0$, except $c(0,0) = 1$

Problem 2.1. Before we discuss the general proof, let's try to figure out what is going on here. Let $k = 2$ and $n = 4$. We want to use the number of permutations in S_3 with one or two cycles in order to count the number of permutations in S_4 with exactly two cycles. How do you take a permutation in S_3 and make it into a permutation in S_4? Note: $c(4,2) = 11$.

Hint for general proof: where do we want the element n to go? A cycle by itself, or a cycle with other elements? How can you use the permutations in S_{n-1} with $k-1$ or k cycles to build this recursion?

Definition 2.2. Let x be a variable. A sequence $(a_0, a_1, a_2 \ldots, a_n)$ of complex numbers has **ordinary generating polynomial**

$$f(x) = a_0 + a_1 x^1 + \ldots + a_n x^n = \sum_{k=0}^{n} a_k x^k$$

Theorem 2.1. For $n \geq 2$, the generating function for the number of permutations σ with $inv(\sigma) = k$, denoted a_k can be written as

$$\sum_{k=0}^{n} a_k q^k = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)}$$

and furthermore, this generating polynomial will factor as

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} = (1+q)(1+q+q^2)\ldots(1+q+\ldots+q^{n-1})$$

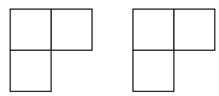
Fact you can use: you will show that the Lehmer code is such that $\mathrm{inv}(\sigma) = \sum_{i \in [n]} L(\sigma_i)$.

Definition 2.3. Let λ be an integer partition of n (written $\lambda \vdash n$) into k parts, such that the sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ is non-decreasing and $\lambda_1 + \ldots + \lambda_k = n$. A **standard Young tableau of shape λ** is a bijective filling of the Young diagram of shape λ with the elements of $[n]$ so that each row and column is weakly increasing. The number of standard Young tableaux of shape λ is denoted f^λ

Example 2.1. Below is a standard Young tableau of shape $\lambda = (3, 3, 1)$.

1	2	5
3	6	7
4		

Find the two standard Young tableau of shape $\lambda = (2, 1)$:



Theorem 2.2. For any given $n \geq 0$, we have

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

The key to the proof of this theorem is constructing a bijection between S_n and pairs of standard young tableau. First we need the bumping algorithm defined below:

1. Set $R :=$ the first row of diagram P
2. While x is less than some element of row R, let y be the leftmost such element. Replace y with x in R. Repeat this step with $R :=$ the row below R, and $x := y$.
3. Now x is greater than every element of R, so place x at the end of this row and terminate.

The first tableau is constructed using the bumping algorithm that will take all the entries of $\sigma = \sigma_1 \ldots \sigma_n$ from left to right, while the second tableau will record the order in which the boxes were added to P.

Problem 2.2. Consider $\sigma = 15423$.

3 Matrix Representations of Permutations

Every permutation can be represented as a matrix filled only with 0's and 1's.

Definition 3.1. For $\sigma \in S_n$, we define the $n \times n$ matrix A_σ such that

$$(A_\sigma)_{ij} = \begin{cases} 1 & \text{if } \sigma_i = j \\ 0 & \text{else} \end{cases}$$

Example 3.1. Let's start with a small set that is still interesting: S_3. First, list all six elements in this set in their one line notations $\sigma = \sigma_1\sigma_2\sigma_3$. Then, below each function, write the 3×3 matrix that represents that permutation.

Problem 3.1. Consider how the definition of the matrix representation of a permutation could give us an alternate proof that there are $n!$ permutations.

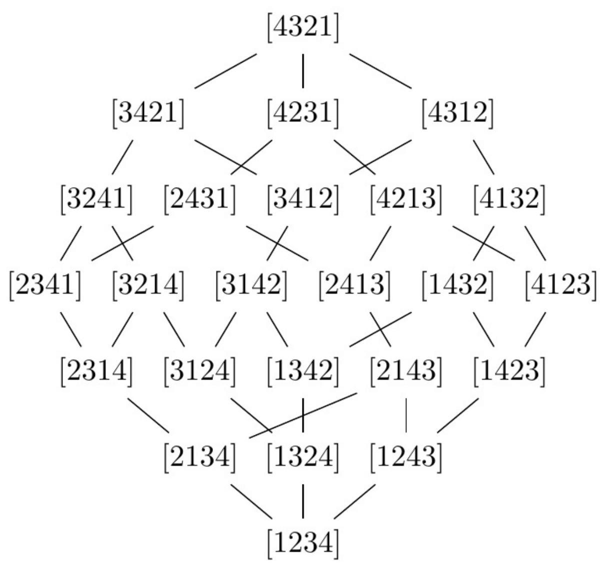
4 Pattern Avoidance

Studying permutations in their matrix form can tell us about other types of objects.

Definition 4.1. Let $\sigma = [a_1 \dots a_n]$, and $p = p_1 \dots p_m$, for $m \leq n$ and $p_i \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq m$. The permutation σ **contains the pattern p** if there exist indices $i_1 < \dots < i_m$ such that $a_{i_1} \dots a_{i_m}$ are in the same order as $p_1 \dots p_m$. That is, $p_j < p_k$ if and only if $a_{i_j} < a_{i_k}$. If σ does not contain p, then σ is **p pattern avoiding**.

For example, $\sigma = [321]$ is 312 pattern avoiding, while the permutation $\omega = [53421]$ contains the pattern 312.

Example 4.1. Let's look at S_4 again:



Let's consider the pattern 132. Using the numbers 1-4, in how many ways can we write this pattern?

Now let's sort S_4 into permutations that contain the pattern, and those that do not.

Definition 4.2. For $\sigma \in S_n$, we define the $n \times n$ matrix A_σ such that

$$(A_\sigma)_{ij} = \begin{cases} 1 & \text{if } \sigma_i = j \\ 0 & \text{else} \end{cases}$$

Example 4.2. Let's look at the matrices associated with the permutations in S_2 and S_3. What properties do you notice about these matrices?

We can take these matrices and construc objects called **Lattice Paths**.

Example 4.3. Let's look at the lattice paths associated with the permutations in S_2 and S_3.

Do you recognize any of these lattice paths? What do you notice about the path associated to the permutation $\sigma = 321$?

Problem 4.1. Using a different rule to take a permutation matrix and draw a lattice path, discuss how we could find a bijection between the 132 avoiding permutations and the **Catalan numbers**.

Directions for Investigation

Anything we do will be new. Most mathematicians do not study permutation properties by looking at the matricies. If they study matricies, they study something called representation theory. This means that our investigation could produce some really interesting results without too much background, and hopefully not too much effort.

- As mentioned, exploring permutation statistics with matrices is something I would like us to think about for a bit. This direction could lead to a expository publication.
- I have also worked extensively with permutation patterns, graphs based off permutations, inverse permutations constructed from products in larger S_n's, etc. Free papers here if you are curious.
- Additionally, if we come up with something in our discussions and you want to play with something that does not involve matrices, we can do that.

References

[1] Jeremy Martin, Lecture Notes on Algebraic Combinatorics, (2018).

[2] Bruce Sagan, *Combinatorics: The Art of Counting* (2020).

[3] Stanley, R.P., *Enumerative Combinatorics, Vol. 1* (2nd Ed.) (2011).

[4] Elder, J., Lafrenière, N., McNicholas, E., Striker, J., Welch, A. preprint available on the arXiv (2022).