Finding Needles in Haystacks

Jennifer Elder

AMS Fall Central Sectional Meeting
October 7, 2023

Collaborators

The work presented is in collaboration with

Pamela E. Harris

University of Wisconsin Milwaukee

Jan Kretschmann

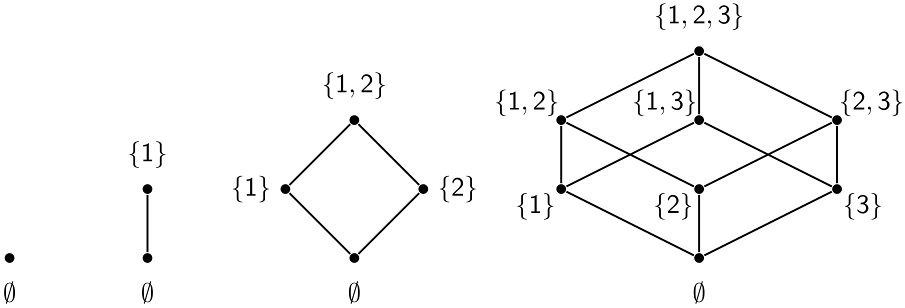
University of Wisconsin Milwaukee

J. Carlos Matínez Mori

SLMath / Georgia Tech

Our needles:

Boolean posets are constructed by subsets of a set I ordered by inclusion.



Boolean posets B_0, B_1, B_2, and B_3 of rank 0, 1, 2, and 3, respectively.
(The rank function is the size of the set.)

Our haystack:

Let $\mathfrak{S}_n$ denote the symmetric group and let s_i denote the simple adjacent transposition switching i and $i+1$.

The *weak right (Bruhat) order*, denoted $W(\mathfrak{S}_n)$, is a poset on $\mathfrak{S}_n$.

Cover relations are defined by applying a single s_i on the right:

Our haystack:

Let $\mathfrak{S}_n$ denote the symmetric group and let s_i denote the simple adjacent transposition switching i and $i+1$.

The *weak right (Bruhat) order*, denoted $W(\mathfrak{S}_n)$, is a poset on $\mathfrak{S}_n$.

Cover relations are defined by applying a single s_i on the right:

- $\tau \lessdot \sigma$ if and only if $\tau s_i = \sigma$ for some

$$i \in \mathrm{Des}(\sigma) = \{j \in [n-1] \ : \ \sigma_j > \sigma_{j+1}\}.$$

- $\tau \lessdot \sigma$ if and only if $\tau s_i = \sigma$ for some

$$i \in \mathrm{Asc}(\tau) = \{j \in [n-1] \ : \ \sigma_j < \sigma_{j+1}\}.$$

Our haystack:

Let $\mathfrak{S}_n$ denote the symmetric group and let s_i denote the simple adjacent transposition switching i and $i+1$.

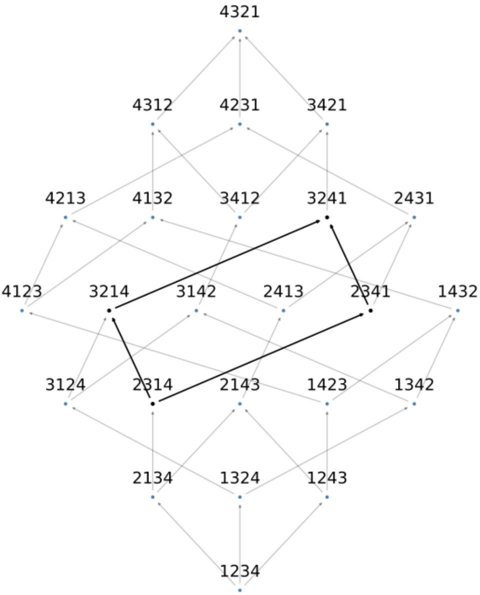
The *weak right (Bruhat) order*, denoted $W(\mathfrak{S}_n)$, is a poset on $\mathfrak{S}_n$.

Cover relations are defined by applying a single s_i on the right:

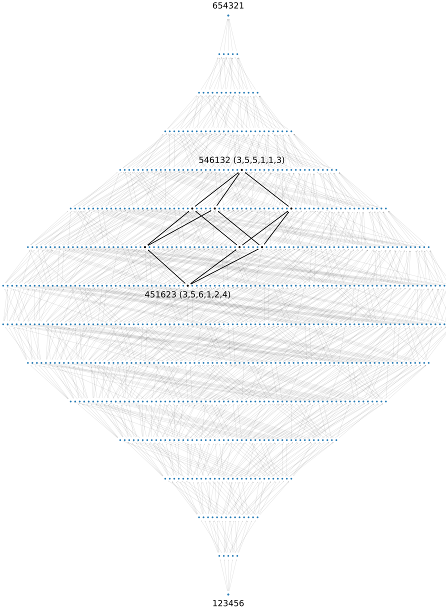
- $\tau \lessdot \sigma$ if and only if $\tau s_i = \sigma$ for some
$$i \in \mathrm{Des}(\sigma) = \{j \in [n-1] \ : \ \sigma_j > \sigma_{j+1}\}.$$
- $\tau \lessdot \sigma$ if and only if $\tau s_i = \sigma$ for some
$$i \in \mathrm{Asc}(\tau) = \{j \in [n-1] \ : \ \sigma_j < \sigma_{j+1}\}.$$

In general, if $\tau \leq \sigma$, then there exists a collection $s_{i_1}, \ldots, s_{i_k}$ of simple transpositions such that
$$\tau s_{i_1} \ldots s_{i_k} = \sigma.$$

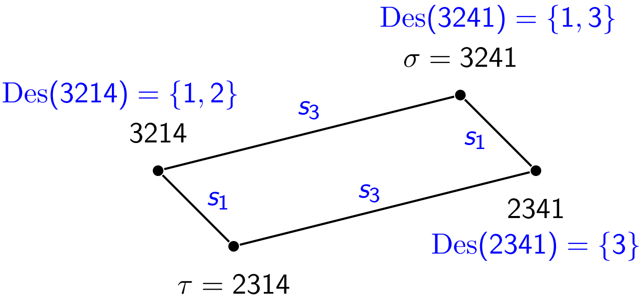
Our needle and our haystack



Our needle and our haystack



Example of cover relations in $W(\mathfrak{S}_4)$



$$\sigma = \tau s_1 s_3$$

Boolean intervals in $W(\mathfrak{S}_n)$

A subset $[\tau, \sigma] \subseteq W(\mathfrak{S}_n)$ is an interval if $\tau \leq \sigma$ and $\pi \in [\tau, \sigma]$ whenever $\tau \leq \pi \leq \sigma$.

Boolean intervals in $W(\mathfrak{S}_n)$

A subset $[\tau, \sigma] \subseteq W(\mathfrak{S}_n)$ is an interval if $\tau \leq \sigma$ and $\pi \in [\tau, \sigma]$ whenever $\tau \leq \pi \leq \sigma$.

Results of Bridget E. Tenner appearing in *Interval structures in the Bruhat and weak orders.* J. Comb., 13(1):135 – 165, 2022.

Corollary 4.4. Boolean posets appear as intervals $[v, w]$ in $W(\mathfrak{S}_n)$ if and only if $v^{-1}w$ is a permutation composed of only commuting generators.

Boolean intervals in $W(\mathfrak{S}_n)$

A subset $[\tau, \sigma] \subseteq W(\mathfrak{S}_n)$ is an interval if $\tau \leq \sigma$ and $\pi \in [\tau, \sigma]$ whenever $\tau \leq \pi \leq \sigma$.

Results of Bridget E. Tenner appearing in *Interval structures in the Bruhat and weak orders.* J. Comb., 13(1):135 – 165, 2022.

Corollary 4.4. Boolean posets appear as intervals $[v, w]$ in $W(\mathfrak{S}_n)$ if and only if $v^{-1}w$ is a permutation composed of only commuting generators.

Proposition 5.9. Let $i \in [n-1]$ be fixed. The number of Boolean intervals in $W(\mathfrak{S}_n)$ of the form $[s_i, w]$ is $F_{i+1}F_{n-i+1}$, where F_i is the ith the ~~Fibonacci~~ Pingala number.

Boolean intervals in $W(\mathfrak{S}_n)$

From Exercise 3.185(h) in *Enumerative Combinatorics Vol. 1*, by setting $q = 1$ into the exponential generating function

$$F(x,q) = \sum_{n\geq 0}\sum_{k\geq 0} f(n,k)q^k \frac{x^n}{n!} = \frac{1}{1 - x - \frac{q}{2}x^2}, \tag{1}$$

Stanley points out that the *total* number of Boolean intervals in $W(\mathfrak{S}_n)$ (OEIS A080599) satisfies the recurrence relation

$$f(n+1) = (n+1)f(n) + \binom{n+1}{2} f(n-1), \tag{2}$$

where $f(0) = 1$ and $f(1) = 1$.

How Do You Find Needles in a Haystack?

Question: If you pick a minimal element $\tau \in W(\mathfrak{S}_n)$, how many Boolean intervals are of the form $[\tau, \sigma]$?

Question: How many Boolean intervals of rank $k \leq n$ exist in $W(\mathfrak{S}_n)$?

Question: Can all Boolean intervals with some minimal element $\tau \in W(\mathfrak{S}_n)$ be enumerated with products of ~~Fibonacci~~ Pingala numbers?

All of the mathematical projects were related to parking functions!

A scenic detour before returning to our needles in haystacks!
(AI generated image with prompt “cars parked along a street”)

Unit interval parking functions

Unit interval parking functions are a subset of parking functions in which cars park exactly at their preferred spot or one spot away.

Example:

- $\alpha = (1, 2, 3, 4, 5)$ is a unit interval parking function and outcome (order in which cars park) is the permutation $\mathcal{O}(\alpha) =$12345.

Unit interval parking functions

Unit interval parking functions are a subset of parking functions in which cars park exactly at their preferred spot or one spot away.

Example:

- $\alpha = (1, 2, 3, 4, 5)$ is a unit interval parking function and outcome (order in which cars park) is the permutation $\mathcal{O}(\alpha) =$12345.
- $\alpha = (5, 1, 1, 4, 3)$ is a unit interval parking function with outcome

2	3	5	4	1
1	2	3	4	5

Unit interval parking functions

Unit interval parking functions are a subset of parking functions in which cars park exactly at their preferred spot or one spot away.

Example:

- $\alpha = (1, 2, 3, 4, 5)$ is a unit interval parking function and outcome (order in which cars park) is the permutation $\mathcal{O}(\alpha) =$12345.
- $\alpha = (5, 1, 1, 4, 3)$ is a unit interval parking function with outcome

$$\begin{array}{ccccc} \underline{2} & \underline{3} & \underline{5} & \underline{4} & \underline{1} \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

- $\alpha = (2, 1, 2, 4, 5)$ is a unit interval parking function with outcome $\mathcal{O}(\alpha) =$21345.

Unit interval parking functions

Unit interval parking functions are a subset of parking functions in which cars park exactly at their preferred spot or one spot away.

Example:

- $\alpha = (1, 2, 3, 4, 5)$ is a unit interval parking function and outcome (order in which cars park) is the permutation $\mathcal{O}(\alpha) =$12345.

- $\alpha = (5, 1, 1, 4, 3)$ is a unit interval parking function with outcome

$$\begin{array}{ccccc} \underline{2} & \underline{3} & \underline{5} & \underline{4} & \underline{1} \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

- $\alpha = (2, 1, 2, 4, 5)$ is a unit interval parking function with outcome $\mathcal{O}(\alpha) =$21345.

- $(1, 1, 1, 1, 1)$ is a parking function but not a unit interval parking function.

Unit interval parking functions

Let UPF_n denote the set of unit interval parking functions of length n.

- 2021: Dr. Shanise Walker poses the question "What if cars can only park in their preference or one away?"

Unit interval parking functions

Let UPF_n denote the set of unit interval parking functions of length n.

- 2021: Dr. Shanise Walker poses the question "What if cars can only park in their preference or one away?"
- 2021: Hadaway and Harris prove that $|\mathsf{UPF}_n| = \mathrm{Fub}_n$, where Fub_n are the Fubini numbers, also known as the ordered Bell numbers (OEIS A000670).

Unit interval parking functions

Let UPF_n denote the set of unit interval parking functions of length n.

- 2021: Dr. Shanise Walker poses the question "What if cars can only park in their preference or one away?"
- 2021: Hadaway and Harris prove that $|\mathsf{UPF}_n| = \mathrm{Fub}_n$, where Fub_n are the Fubini numbers, also known as the ordered Bell numbers (OEIS A000670).
- 2022: Bradt, E., Harris, Rojas Kirby, Reutercrona, Wang, and Whidden provided a direct bijection between unit interval parking functions and objects called Fubini rankings. This bijection provided insights on: the number of ways to permute the entires of a unit interval parking function and preserve the unit property, a new formula for the Fubini numbers, and generalizations to r-Fubini numbers.

Unit interval parking functions

Let UPF_n denote the set of unit interval parking functions of length n.

- 2021: Dr. Shanise Walker poses the question "What if cars can only park in their preference or one away?"
- 2021: Hadaway and Harris prove that $|\mathrm{UPF}_n| = \mathrm{Fub}_n$, where Fub_n are the Fubini numbers, also known as the ordered Bell numbers (OEIS A000670).
- 2022: Bradt, E., Harris, Rojas Kirby, Reutercrona, Wang, and Whidden provided a direct bijection between unit interval parking functions and objects called Fubini rankings. This bijection provided insights on: the number of ways to permute the entires of a unit interval parking function and preserve the unit property, a new formula for the Fubini numbers, and generalizations to r-Fubini numbers.
- 2023: E., Harris, Martìnez Mori, et al. have further generalizations to intervals of length ℓ for traditional and generalized parking functions.

Fubini Rankings

Definition

A Fubini ranking of length n is a tuple $r = (r_1, r_2, \ldots, r_n) \in [n]^n$ that records a valid ranking over n competitors with ties allowed (i.e., multiple competitors can be tied and have the same rank).

Note, if k competitors are tied and rank ith, the $k-1$ subsequent ranks $i+1, i+2, \ldots, i+k-1$ are disallowed.

Fubini Rankings

Definition

A Fubini ranking of length n is a tuple $r = (r_1, r_2, \ldots, r_n) \in [n]^n$ that records a valid ranking over n competitors with ties allowed (i.e., multiple competitors can be tied and have the same rank).

Note, if k competitors are tied and rank ith, the $k-1$ subsequent ranks $i+1, i+2, \ldots, i+k-1$ are disallowed.

Are the following Fubini rankings?

- $(1, 2, 3, 4, 5)$
- $(1, 1, 1, 1, 1)$
- $(3, 1, 5, 1, 3)$
- $(3, 1, 5, 1, 2)$

Fubini rankings and unit interval parking functions

Example:

- $(1, 1, 2)$ is a unit interval parking function but not a Fubini ranking.

Example:

- $(1, 1, 2)$ is a unit interval parking function but not a Fubini ranking. The tie for rank 1 would disallow rank 2.

Fubini rankings and unit interval parking functions

Example:

- $(1, 1, 2)$ is a unit interval parking function but not a Fubini ranking. The tie for rank 1 would disallow rank 2.
- $(1, 1, 1)$ is a Fubini ranking but not a unit interval parking function.

Fubini rankings and unit interval parking functions

Example:

- $(1, 1, 2)$ is a unit interval parking function but not a Fubini ranking. The tie for rank 1 would disallow rank 2.
- $(1, 1, 1)$ is a Fubini ranking but not a unit interval parking function. The third car would park in spot 3, which is not at most one away from its preference for parking spot 1.

Fubini rankings and unit interval parking functions

Example:

- $(1, 1, 2)$ is a unit interval parking function but not a Fubini ranking.
  The tie for rank 1 would disallow rank 2.

- $(1, 1, 1)$ is a Fubini ranking but not a unit interval parking function.
  The third car would park in spot 3, which is not at most one away from its preference for parking spot 1.

- $(1, 3, 1)$ is both a Fubini ranking and a unit interval parking function.

Fubini rankings and unit interval parking functions

Example:

- $(1, 1, 2)$ is a unit interval parking function but not a Fubini ranking. The tie for rank 1 would disallow rank 2.
- $(1, 1, 1)$ is a Fubini ranking but not a unit interval parking function. The third car would park in spot 3, which is not at most one away from its preference for parking spot 1.
- $(1, 3, 1)$ is both a Fubini ranking and a unit interval parking function.

Unit Fubini rankings are the elements in

$$\mathsf{UFR}_n = \mathsf{FR}_n \cap \mathsf{UPF}_n,$$

which are Fubini rankings with at most two competitors tying for any one rank.

Unit Fubini Rankings

Theorem (E.-Harris- Kretschmann- Martínez Mori, 2023)

The number of Unit Fubini rankings of length n are enumerated by OEIS A080599, satisfying the recurrence relation

$$f(n+1) = (n+1)f(n) + \binom{n+1}{2} f(n-1), \tag{3}$$

where $f(0) = 1$ and $f(1) = 1$.

Unit Fubini Rankings

Theorem (E.-Harris- Kretschmann- Martínez Mori, 2023)

The number of Unit Fubini rankings of length n are enumerated by OEIS A080599, satisfying the recurrence relation

$$f(n+1) = (n+1)f(n) + \binom{n+1}{2} f(n-1), \tag{3}$$

where $f(0) = 1$ and $f(1) = 1$.

Theorem (E.-Harris- Kretschmann- Martínez Mori, 2023)

The number of unit Fubini rankings with $n-k$ distinct ranks is given by

$$\frac{n!}{2^k}\binom{n-k}{k}.$$

So what's the big idea now?

We now connect unit Fubini rankings with Boolean intervals in the weak order of $\mathfrak{S}_n$.

So what's the big idea now?

We now connect unit Fubini rankings with Boolean intervals in the weak order of $\mathfrak{S}_n$.

Theorem (E.-Harris- Kretschmann- Martínez Mori, 2023)

The set of unit Fubini rankings with $n - k$ distinct ranks is in bijection with the set of Boolean intervals in $W(\mathfrak{S}_n)$ of rank k.

So what's the big idea now?

We now connect unit Fubini rankings with Boolean intervals in the weak order of $\mathfrak{S}_n$.

Theorem (E.-Harris- Kretschmann- Martínez Mori, 2023)

The set of unit Fubini rankings with $n-k$ distinct ranks is in bijection with the set of Boolean intervals in $W(\mathfrak{S}_n)$ of rank k.

Corollary (E.-Harris- Kretschmann- Martínez Mori, 2023)

The number of Boolean intervals in $W(\mathfrak{S}_n)$ of rank k is given by

$$f(n,k) = \frac{n!}{2^k}\binom{n-k}{k}.$$

Our Bijection: Φ

- First, choose a permutation in the weak order: $\tau = 412356$. Note that $\mathrm{Asc}(\tau) = \{2, 3, 4, 5\}$.

Our Bijection: Φ

- First, choose a permutation in the weak order: $\tau = 412356$. Note that $\mathrm{Asc}(\tau) = \{2, 3, 4, 5\}$.
- Next, we want to find the permutation whose outcome is τ when we treat it as a parking function: $\alpha = \tau^{-1} = 234156$.

Our Bijection: Φ

- First, choose a permutation in the weak order: $\tau = 412356$. Note that $\mathrm{Asc}(\tau) = \{2, 3, 4, 5\}$.
- Next, we want to find the permutation whose outcome is τ when we treat it as a parking function: $\alpha = \tau^{-1} = 234156$.
- We observe that the only possible subsets of $\mathrm{Asc}(\tau) = \{2, 3, 4, 5\}$ consisting of nonconsecutive integers are: $\emptyset$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{2, 4\}$, $\{2, 5\}$, and $\{3, 5\}$.

Our Bijection: Φ

- First, choose a permutation in the weak order: $\tau = 412356$. Note that $\mathrm{Asc}(\tau) = \{2, 3, 4, 5\}$.
- Next, we want to find the permutation whose outcome is τ when we treat it as a parking function: $\alpha = \tau^{-1} = 234156$.
- We observe that the only possible subsets of $\mathrm{Asc}(\tau) = \{2, 3, 4, 5\}$ consisting of nonconsecutive integers are: $\emptyset$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{2, 4\}$, $\{2, 5\}$, and $\{3, 5\}$.
- We use these subsets to create all the unit interval parking functions with outcome τ:

$$\begin{array}{ccc} \delta_{\emptyset}(\alpha) = 234156, & \delta_{\{2\}}(\alpha) = 224156, & \delta_{\{3\}}(\alpha) = 233156 \\ \delta_{\{4\}}(\alpha) = 234146 & \delta_{\{5\}}(\alpha) = 234155, & \delta_{\{2,4\}}(\alpha) = 224146, \\ \delta_{\{2,5\}}(\alpha) = 224155, & \delta_{\{3,5\}}(\alpha) = 233155. & \end{array}$$

Our Bijection: Φ

For each $\pi \in \mathcal{O}(\alpha)$, we place the label $\Phi(\pi)$ at the top of the Boolean interval it is mapped to.

$$\tau = 412356 = \Phi(234156)$$

Our Bijection: Φ

For each $\pi \in \mathcal{O}(\alpha)$, we place the label $\Phi(\pi)$ at the top of the Boolean interval it is mapped to.

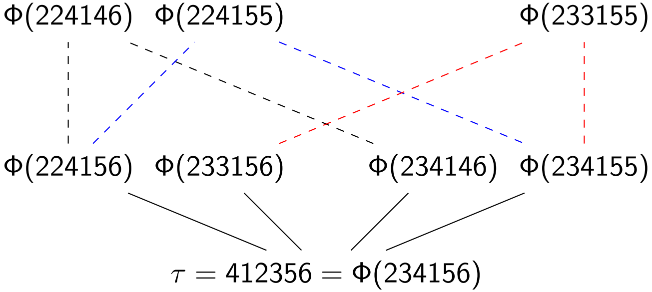
$$\Phi(224156) \quad \Phi(233156) \qquad \Phi(234146) \quad \Phi(234155)$$

$$\tau = 412356 = \Phi(234156)$$

Note that, left to right, the solid edges represent the application of s_2, s_3, s_4, and s_5 to τ.

Our Bijection: Φ

For each $\pi \in \mathcal{O}(\alpha)$, we place the label $\Phi(\pi)$ at the top of the Boolean interval it is mapped to.



Note that, left to right, the solid edges represent the application of s_2, s_3, s_4, and s_5 to τ.

Product of the ~~Fibonacci~~ Pingala Numbers

Theorem (E.-Harris- Kretschmann- Martínez Mori, 2023)

Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ be in one-line notation and partition its ascent set $\mathrm{Asc}(\pi) = \{i \in [n-1] : \pi_i < \pi_{i+1}\}$ into maximal blocks $b_1, b_2, \ldots, b_k$ of consecutive entries. Then, the number of Boolean intervals $[\pi, w]$ in $W(\mathfrak{S}_n)$ with fixed minimal element π and arbitrary maximal element w (including the case $\pi = w$) is given by

$$\prod_{i=1}^{k} F_{|b_i|+2},$$

where F_ℓ is the ℓth the ~~Fibonacci~~ Pingala number, and $F_1 = F_2 = 1$.

Future Work

- How can we utilize unit Fubini rankings, or a slight generalization thereof, to enumerate Boolean intervals in Bruhat and weak orders of other Coxeter systems?
- In particular we ask: How many Boolean intervals are there in the weak order of the hyperoctahedral group (type B Coxeter group) with minimal element π?

Thank you!

Email: jelder8@missouriwestern.edu

arXiv Preprint